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Principle Conditioning

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The definition of the conditional probability is very important in the theory of the probability. This definition is based on the fact, that random events can be simultaneously measurable. This paper deals with the problem of conditioning for such random events, which are not simultaneously measurable. This paper defines conditional states as convex combination of special states.

KEY WORDS: quantum logic; orthomodeular lattice; state; conditional probability; independence.

1. INTRODUCTION

The classical Kolmogorovian model for random events was developed only for such random events, which are simultaneouly measurable (in another words, which are compatible). The basic algebraic structure, which is used as a model for noncompatible random events is an orthomodular lattice (OML), or an orhtomodular σ-lattice (σ-OML). In this paper we determine a conditional state (analogical notion of conditional probability) as a convex combination of "orthogonal states" on an OML.

In the classical theory we assume that random events can be interpreted as a set of outcomes of experiments. A probability space is a triple (Kolmogoroff, 1933; Renyi, 1947, 1955) $(\Omega, \mathcal{B}, P)$, where Ω is a set of all elementary random events, $\mathcal{B}$ is a σ-algebra of subset of Ω and P is a probability measure. In the noncommutative approach we have a couple (L, M), where L is a σ-OML and M is a set of states on it.

Let $(\Omega_i, \mathcal{F}_i)$ for $i = 1, \ldots, n$ be measurable spaces. Let $\overline{\Omega} = \Omega_1 \times \ldots \times \Omega_n$. If $\overline{\omega} = (\omega_1, \ldots, \omega_n)$, then $\pi_i(\overline{\omega}) = \omega_i$. Then $L = \{\pi_i^{-1}(A); \quad A \in \mathcal{F}, \quad i = 1, \ldots, n\}$, where for example $\pi_1^{-1}(A) = (A, \Omega, \ldots, \Omega)$, for $A \in \mathcal{F}_1$. Then L can be organized as an OML by the following way:

(1) $\pi_i^{-1}(\Omega) := 1$;

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(2) $\pi_i^{-1}(A) \vee \pi_i^{-1}(B) := \pi_i^{-1}(A \cup B)$ and $\pi_i^{-1}(A) \vee \pi_j^{-1}(B) := 1$, for $i \neq j$;
(3) $\pi_i^{-1}(\emptyset) := 0$;
(4) $\pi_i^{-1}(A) \perp \pi_j^{-1}(B)$ if and only if $i = j$ and $A \cap B = \emptyset$.

Let $(\Omega_i, \mathcal{F}_i, P_i)$ for $i = 1, \ldots, n$ be some probability spaces and L be an OML defined as before. A map

$$m : L \to [0, 1],$$

such that

$$m\big(\pi_i^{-1}(A)\big) = P_i(A) \quad \text{for each} \quad A \in \mathcal{F}_i$$

is a state on L.

Well known examples of OMLs include Boolean algebras and the orthogonal projections on a Hilbert space.

Definition 1.1: (Varadarajan, 1968). Let L be a nonempty set endowed with a partial order $\leq$ with the largest element (1) and the smallest element (0). Let there be defined the operations of supremum ($\vee$), infimum ($\wedge$) (the lattice operations) and a map $\perp$: $L \to L$ with the following properties:

(i) For any $\{a_n\}_{n\in\mathcal{A}} \in L$, where $\mathcal{A} \subset N$ is finite ($\mathcal{A}$ is infinite)

$$\bigvee_{n\in\mathcal{A}} a_n, \bigwedge_{n\in\mathcal{A}} a_n \in L.$$

(ii) For any $a \in L$ $(a^\perp)^\perp = a$.
(iii) If $a \in L$, then $a \vee a^\perp = 1$.
(iv) If $a, b \in L$ such that $a \leq b$, then $b^\perp \leq a^\perp$.
(v) If $a, b \in L$ such that $a \leq b$ then $b = a \vee (a^\perp \wedge b)$ (orthomodular law).

Then $(L, 0, 1, \vee, \wedge, \perp)$ is called *an orthomodular lattice* (briefly *L is an OML*) (*a σ-OML*).

Let L be an OML (a σ-OML). Then the elements $a, b \in L$ will be called:

(1) *orthogonal* $(a \perp b)$ iff $a \leq b^\perp$;
(2) *compatible* $(a \leftrightarrow b)$ iff there exist mutually orthogonal elements a_1, b_1, $c \in L$ such that

$$a = a_1 \vee c \quad \text{and} \quad b = b_1 \vee c.$$

If $a_i \in L$ for any $i \in \mathcal{A}$ and $b \in L$ is such, that $b \leftrightarrow a_i$ for all i, then $b \leftrightarrow \bigvee_{i\in\mathcal{A}} a_i$ and

$$b \wedge \bigvee_{i\in\mathcal{A}} a_i = \bigvee_{i\in\mathcal{A}} a_i \wedge b$$

(Dvurečenskij and Pulmannová, 2000).

Definition 1.2: (Varadarajan, 1968) A map $m : L \to R$ such that

(i) $m(0) = 0$ and $m(1) = 1$,
(ii) if $a \perp b$ then $m(a \vee b) = m(a) + m(b)$,

is called *a state* on L. If L is a σ-OML and m is a σ-additive function then m will be called a σ-*state*.

2. A CONDITIONAL STATE ON AN OML

Definition 2.1. Let L be an OML. A subset $L_0 \subset L - \{0\}$ is called a conditional system (CS) *(a σ-CS)* if the following conditions are fulfilled:

(1) If $a, b \in L_0$, then $a \vee b \in L_0$. (If $a_n \in L_0$, for $n = 1, 2, \ldots$, then $\bigvee_n a_n \in L_0$.)
(2) If $a, b \in L_0$ and $a < b$, then $a^\perp \wedge b \in L_0$.

Definition 2.2. Let L be an OML and L_0 be a CS (a σ-CS). Let

$$f : L \times L_0 \to [0, 1].$$

If the function f fulfils the following conditions:

(C1) for each $a \in L_0$ $f(., a)$ is a state on L (a σ-state);
(C2) for each $a \in L_0$ $f(a, a) = 1$;
(C3) if $\{a_n\}_{n \in \mathcal{A}} \in L_0$, where $\mathcal{A} \subset N$, $\mathcal{A}$ has finite cardinality ($\mathcal{A}$ can be infinite), and a_n are mutually orthogonal, then for each $b \in L$

$$f(b, \bigvee_{n \in \mathcal{A}} a_n) = \sum_{n \in \mathcal{A}} f(a_n, \bigvee_{n \in \mathcal{A}} a_n) f(b, a_n);$$

then f is called *a conditional state* (*a σ-conditional state).*

It is clear, that if L is a σ-OML, $\{a_i\}_{i \in \mathcal{A}}$, where $\mathcal{A} \subset N$, such that $a_i \perp a_j$, for $i \neq j$, than we can rewrite the Proposition 1.1 for a σ-conditional state. Moreover for any $\{a_i\}_{i \in \mathcal{A}}$ there exists many conditional states (or σ-conditional states). On the other hand, because a measurable space can be described as σ-OML (Varadarajan, 1968), then this representation is fulfilled for a probability space, too.

It is clear, that if there exists a probability measure μ on the measurable space $(\Omega, \mathcal{B})$, then the conditional probability f exists on $\mathcal{B} \times \mathcal{B}_0$ and

$$f(A, B) = \frac{\mu(A \cap B)}{\mu(B)},$$

where $\mathcal{B}_0 \subset \{E \in \mathcal{B}; \quad \mu(E) \neq 0\}$. The system $(\Omega, \mathcal{B}, \mathcal{B}_0, f)$ is called the condional probabilty system (CPS).

Let $\mathcal{P}$ be some collection of probability measures on $(\Omega, \mathcal{B})$. It is a question, when this collection $\mathcal{P}$ can be organized as a system of conditional probabilities. On the classical theory of probability the following theorems are fulfiled:

Proposition 2.1. *Let $(\Omega, \mathcal{B}, \mathcal{B}_0, f)$ be a CPS. Let $\{B_i\}_{i\in\mathcal{A}} \in \mathcal{B}_0$, $\mathcal{A} \subset N$ and let there exist $B \in \mathcal{B}_0$, such that $f(B, B_i) = 1$ and $f(B_i, B) > 0$ for any $i \in \mathcal{A}$. Then for any $C \in \mathcal{B}$*

$$f(C, B) = \sum_{i\in\mathcal{A}} f(C, B_i) f(B_i, B)$$

iff

$$f\Big(\bigcup_{i\in\mathcal{A}} B_i, B\Big) = \sum_{i\in\mathcal{A}} f(B_i, B) = 1.$$

Proposition 2.2. *Let $(\Omega, \mathcal{B}, \mathcal{B}_0, f)$ be a CPS. Let $\{B_i\}_{i\in\mathcal{A}} \in \mathcal{B}_0$, $\mathcal{A} \subset N$ and let there exist $B \in \mathcal{B}_0$, such that $f(B, B_i) = 1$ and $f(B_i, B) > 0$ for any $i \in \mathcal{A}$. Then for any $C \in \mathcal{B}$*

$$f(C, B) = \sum_{i\in\mathcal{A}} f(C, B_i) f(B_i, B),$$

then for any $i \neq j$ $f(B_i, B_j) = 0$.

From this aproach follows, that the definition of a conditional state (a σ-conditional state) on an OML (a σ-OML) has been defined correctly. More details about the classical aproach to the conditional probability we can find for example in (Riečcan and Neubrun, 1997).

Proposition 2.3. *Let L be an OML. Let $\{a_i\}_{i=1}^n \in L$, $n \in N$ where $a_i \perp a_j$ for $i \neq j$. Let for any i there exists a state α_i, such that $\alpha_i(a_i) = 1$. Then there exists a CS such that for any $\mathbf{k} = (k_1, k_2, \ldots, k_n)$, where $k_i \in [0; 1]$ for $i \in \{1, 2, \ldots, n\}$ with the property $\sum_{i=1}^n k_i = 1$, there exists a conditional state*

$$f_{\mathbf{k}} : L \times L_0 \to [0; 1],$$

and

(1) for any i and each $d \in L$ $f_{\mathbf{k}}(d, a_i) = \alpha_i(d)$;
(2) for each a_i

$$f_{\mathbf{k}}\Big(a_i, \bigvee_{i=1}^n a_i\Big) = k_i;$$

Proof: Let

$$L_0 = \{c \in L; \quad c = \bigvee_{i \in \mathcal{A}} a_{i \in \mathcal{A}}, \quad \text{for each} \quad \mathcal{A} \subset \{1, 2, \ldots, n\}\}.$$

Then it is clear that L_0 is a CS and so L_0 exists in L.

From the assumption, we have the set of triples $\{(a_i, \alpha_i, k_i), i = 1, \ldots, n\}$, and from the properties of a CS follows that for each $c \in L_0$ $\{i_1, \ldots, i_s\} \subset \{1, \ldots, n\}$ such that

$$c = \bigvee_{j=1}^{s} a_{i_j} \quad \text{and} \quad \alpha_{i_j}(a_{i_j}) = 1.$$

Let as denote $\mathcal{K}(c) = \sum_{j=1}^{s} k_{i_j}$.

Let $f_{\mathbf{k}} : L \times L_0 \to [0, 1]$ such that for each $d \in L$ and $c \in L_0$

$$f_{\mathbf{k}}(d, c) = \frac{1}{\mathcal{K}(c)} \sum_{j=1}^{s} k_{i_j} \alpha_{i_j}(d).$$

Now we show, that $f_{\mathbf{k}}$ is the conditional state.

(C1) Let $c \in L_0$. Then

$$f_{\mathbf{k}}(1, c) = \frac{1}{\mathcal{K}(c)} \sum_{j=1}^{s} k_{i_j} \alpha_{i_j}(1) = \frac{1}{\mathcal{K}(c)} \sum_{j} k_{i_j} = \frac{\mathcal{K}(c)}{\mathcal{K}(c)} = 1$$

and

$$f_{\mathbf{k}}(0, c) = \frac{1}{\mathcal{K}(c)} \sum_{j} k_{i_j} \alpha_{i_j}(0) = \frac{1}{\mathcal{K}(c)} \sum_{j} k_{i_j}.0 = 0$$

Let $d, b \in L$, such that $d \perp b$. Then

$$f_{\mathbf{k}}(d \vee b, c) = \frac{1}{\mathcal{K}(c)} \sum_{j} k_{i_j} \alpha_{i_j}(d \vee b) = \frac{1}{\mathcal{K}(c)} \left(\sum_{j} k_{i_j} \alpha_{i_j}(d) + \sum_{j} k_{i_j} \alpha_{i_j}(b) \right)$$

$$= \frac{1}{K(c)} \sum_{j} k_{i_j} \alpha_{i_j}(d) + \frac{1}{K(c)} \sum_{j} k_{i_j} \alpha_{i_j}(b) = f(d, c) + f(b, c)$$

So $f_{\mathbf{k}}$ is a state on L.

(C2) It is easy to see, that for each $c \in L_0$

$$f_{\mathbf{k}}(c, c) = 1.$$

(C3) It is enough to show it for two orthogonal elements from L_0. Let c_1, c_2 be such elements from L_0, that

$$c_1 = \bigvee_{i=1}^{n_1} a_i \quad \text{and} \quad c_2 = \bigvee_{i=n_1+1}^{n_2} a_i.$$

Then

$$f_{\mathbf{k}}(c_j, c_1 \vee_j c_2) = \frac{\mathcal{K}(c_j)}{\mathcal{K}(c_1 \vee c_2)}$$

and

$$f_{\mathbf{k}}(d, c_1 \vee c_2) = \frac{1}{\mathcal{K}(c_1 \vee c_2)} \sum_{j=1}^{n_2} k_j \alpha_j(d).$$

From it follows

$$\sum_{j}^{n_2} f_{\mathbf{k}}(c_j, c_1 \vee c_2) f_{\mathbf{k}}(d, c_j)$$

$$= \frac{\mathcal{K}(c_1)}{\mathcal{K}(c_1 \vee c_2)} \frac{1}{\mathcal{K}(c_1)} \sum_{i=1}^{n_1} k_i \alpha_i(d)) + \frac{\mathcal{K}(c_2)}{\mathcal{K}(c_1 \vee c_2)} \frac{1}{\mathcal{K}(c_2)} \sum_{i=n_1+1}^{n_2} k_i \alpha_i(d)$$

$$= \frac{1}{\mathcal{K}(c_1 \vee c_2)} \sum_{i=1}^{n_2} k_i \alpha_i(d) = f_{\mathbf{k}}(d, c_1 \vee c_2).$$

So $f_{\mathbf{k}}$ is the conditional state.

Let $a = \bigvee_{i=1}^{n} a_i$. Then

$$f(., a) = \frac{1}{\mathcal{K}(a)} \sum_i k_i \alpha_i(.) = \sum_i k_i \alpha_i(.),$$

and then for each $i = 1, \ldots, n$

$$f(a_i, a) = k_i.$$

From it follows, that for each $d \in L$ and a_i $i = 1, \ldots, n$

$$f_{\mathbf{k}}(d, a_i) = \frac{1}{\mathcal{K}(a_i)} k_i \alpha_i(d) = \alpha_i(d).$$

It is clear that $f_{\mathbf{k}} : L \times L_0 \to [0, 1]$ is the conditional state with the properties (1) and (2). □

3. DEPENDENCE AND INDEPENDENCE

Definition 3.1. Let L be an OML and f be a conditional state. Let $b \in L, a, c \in L_0$ such that $f(c, a) = 1$. Then b is independent of a, with respect to the state $f(., c)$ $(b \asymp_{f(.,c)} a)$ iff $f(b, c) = f(b, a)$.

The classical definition of independence in a probability space $(\Omega, \mathcal{B}, P)$ is a special case of this definition, because

$$P(A|B) = P(A|\Omega) \quad \text{iff} \quad P(A \cap B|\Omega) = P(A|\Omega)P(B|\Omega).$$

Let L is an OML. Let $a_1, \dots, a_n \in L - \{0\}$, such that $a_i \perp a_j$, for $i \neq j$. Let α_i $i = 1, \dots, n$ be such state, $\alpha_i(a_j) = \delta_{i,j}$, the Kronecker $\delta_{i,j}$ which $= 1$ when $i = j$ and 0 otherwice. Then, for each $k_i \in [0, 1]$ ($i = 1, \dots, n$) such that

$$\sum_{i=1}^{n} k_i = 1 \quad \text{a map} \quad \mu := \sum_{i=1}^{n} k_i \alpha_i = f_\mu\Big(., \bigvee_{i=1}^{n} a_i\Big)$$

is a state and we say that α_i is a conditional state with the condition a_i ($\alpha_i = f_\mu(., a_i)$) and $k_i = \mu(a_i)$. Then for $b \in L$

$$b \asymp_\mu a_i \quad \text{iff} \quad \alpha_i(b) = \mu(b).$$

Proposition 3.1. *Let L is an OML. Let $a_1, \dots, a_n \in L - \{0\}$, such that $a_i \perp a_j$, for $i \neq j$. Let α_i $i = 1, \dots, n$ be a state such that $\alpha_i(a_j) = \delta_{i,j}$. Let $k_i \in [0, 1]$ (i=1,...,n), such that*

$$\sum_{i=1}^{n} k_i = 1 \quad \textit{and} \quad \mu = \sum_{i=1}^{n} k_i \alpha_i.$$

Then

(1) $b \asymp_\mu a_i$, *iff* $b \asymp_\mu \vee_{j \neq i} a_j$;
(2) $b \asymp_\mu a_i$, *iff* $b^\perp \asymp_\mu a_i$.

Proof:

(1) It is enough to show it for $i = 1$. Let $b \asymp_\mu a_1$, then from the definition follows, that

$$\alpha_1(b) = \mu(b)$$

and so

$$\mu(b) = f_\mu\Big(b, \bigvee_{i=1}^{n} a_i\Big) = f_\mu\Big(a_1, \bigvee_{i=1}^{n} a_j\Big) f_\mu(b, a_1))$$

$$+ f_\mu\Big(\bigvee_{j=2}^{n} a_j, \bigvee_{i=1}^{n} a_i\Big) f_\mu\Big(b, \bigvee_{j=2}^{n} a_j\Big)$$

$$\alpha_1(b) = k_1 \alpha_1(b) + f_\mu\Big(\bigvee_{j=2}^{n} a_j, \bigvee_{i=1}^{n} a_i\Big) f_\mu\Big(b, \bigvee_{j=2}^{n} a_j\Big)$$

$$(1 - k_1)\alpha_1(b) = f_\mu\Big(\bigvee_{j=2}^{n} a_j, \bigvee_{i=1}^{n} a_i\Big) f_\mu\Big(b, \bigvee_{j=2}^{n} a_j\Big)$$

$$f_\mu\Big(\bigvee_{j=2}^{n} a_j, \bigvee_{i=1}^{n} a_i\Big)\alpha_1(b) = f_\mu(\bigvee_{j=2}^{n} a_j, \bigvee_{i=1}^{n} a_i) f_\mu\Big(b, \bigvee_{j=2}^{n} a_j\Big)$$

$$\alpha_1(b) = f_\mu(b, a_1) = f_\mu\Big(b, \bigvee_{j=2}^{n} a_j\Big).$$

From it follows that $b \asymp_{f_m u} \vee_{j \neq i} a_j$. The converse implication can be shown analogously.

(2) If $b \asymp_\mu a_i$, then

$$\mu(b^\perp) = 1 - \mu(b) = 1 - \alpha_i(b) = \alpha_i(b^\perp)$$

and so $b^\perp \asymp_\mu a_i$. The converse implication can be shown analogously.

□

Proposition 3.2. *Let L be an OML, L_0 be a CS and $f : L \times L_0 \to [0, 1]$ be a conditional state.*

(1) Let $a^\perp, a, c \in L_0$, $b \in L$ and $f(c, a) = f(c, a^\perp) = 1$. Then $b \asymp_{f(.,c)} a$ iff $b \asymp_{f(.,c)} a^\perp$.

(2) Let $a, c \in L_0$, $b \in L$ and $f(c, a) = 1$. Then $b \asymp_{f(.,c)} a$ iff $b^\perp \asymp_{f(.,c)} a$.

(3) Let $a, c, b \in L_0$, $b \leftrightarrow a$ and $f(c, a) = f(c, b) = 1$, $f(a, b) \neq 0$, $f(b, a) \neq 0$. Then $b \asymp_{f(.,c)} a$ iff $a \asymp_{f(.,c)} b$.

(4) Let $b, c, d \in L_0$, $b \perp d$, $a \in L$ and $f(c, b) = f(c, d) = 1$. Then $a \asymp_{f(.,c)} b$, $a \asymp_{f(.,c)} d$ then $a \asymp_{f(.,c)} b \vee d$

Proof:

(1) From the definition of a conditional state follows, that for each $x \in L$

$$f(x, c) = f(a, c) f(x, a) + f(a^\perp, c) f(x, a^\perp). \tag{1}$$

Let $b \asymp_{f(.,c)} a$. It means, that $f(b, c) = f(b, a)$. If we put $x = b$, then we get

$$f(b, a) = f(b, c) = f(a, c) f(b, a) + f(a^\perp, c) f(b, a^\perp).$$

thus

$$(1 - f(a, c)) f(b, a) = f(a^\perp, c) f(b, a^\perp),$$

but $1 - f(a, c) = f(a^\perp, c)$. Then

$$f(a^\perp, c) f(b, a) = f(a^\perp, c) f(b, a^\perp)$$

and so

$$f(b, a) = f(b, a^\perp) = f(b, c).$$

Thus $b \asymp_{f(.,c)} a$. The converse implication can be shown analogously.

(2) Let $b \asymp_{f(.,c)} a$. Then $f(b,c) = f(b,a)$, and so $1 - f(b,c) = 1 - f(b,a)$. Thus $f(b^{\perp}, c) = f(b^{\perp}, a)$. The converse implication can be shown analogously.

(3) By (1) with $x = a \wedge b$, we have

$$f(b,a) = f(a \wedge b, a). \tag{2}$$

On the other hand, by (1) with b in place a, we have

$$f(a,b) = f(a \wedge b, b).$$

From the definition of a conditional state, we can write

$$f(x,c) = f(a,c)f(x,a) + f(a^{\perp}, c)f(x, a^{\perp}),$$

for each $x \in L$. If we put $x = a \wedge b$, then

$$f(a \wedge b, c) = f(a,c)f(a \wedge b, a) = f(a,c)f(b,a).$$

On the other hand

$$f(x,c) = f(b,c)f(x,b) + f(b^{\perp}, c)f(x, b^{\perp}),$$

and we get

$$f(a \wedge b, c) = f(b,c)f(a \wedge b, b) = f(b,c)f(a,b).$$

But $b \asymp_{f(.,c)} a$, it means $f(b,c) = f(b,a)$. Then

$$f(a \wedge b, c) = f(b,c)f(a,b) = f(b,a)f(a,b)$$

analogously

$$f(a \wedge b, c) = f(a,c)f(b,a),$$

and, by (2) we can write

$$f(b,a)f(a,b) = f(a,c)f(b,a),$$

so that, since $f(a,b) \neq 0$

$$f(a,b) = f(a,c)$$

and $a \asymp_{f(.,c)} b$. The converse implication can be shown analogously.

(4) Let $b \perp d$, $f(c,b) = f(c,d) = 1$. Then

$$\begin{aligned} f(c, b \vee d) &= f(d, d \vee b)f(c,d) + f(b, d \vee b)f(c,b) \\ &= f(d, d \vee b) + f(b, d \vee b) \\ &= f(b \vee d, b \vee d) = 1. \end{aligned}$$

If $a \asymp_{f(.,c)} b$, $a \asymp_{f(.,c)} d$, then $f(a, b) = f(a, c) = f(a, d)$ and

$$\begin{aligned} f(a, b \vee d) &= f(d, d \vee b) f(a, d) + f(b, d \vee b) f(a, b) \\ &= f(d, d \vee b) f(a, c) + f(b, d \vee b) f(a, c) \\ &= f(b \vee d, b \vee d) f(a, c) = f(a, c). \end{aligned}$$

It means $a \asymp_{f(.,c)} b \vee d$. □

Example. Let $L = \{a, a^{\perp}, b, b^{\perp}, 0, 1\}$ and $L_0 = L - \{0\}$. Let α, α' be such states on L, that $\alpha(a) = \alpha'(a^{\perp}) = 1$ and let $\mathbf{k} = (\mathbf{0.1}, \mathbf{0.9})$. Then we can define a conditional state by the following way:

$$\begin{aligned} f_{\mathbf{k}}(d, a) &= \alpha(d) \quad \text{and} \quad f_{\mathbf{k}}(d, a^{\perp}) = \alpha'(d) \\ f_{\mathbf{k}}(d, 1) &= 0.1\alpha(d) + 0.9\alpha'(d) \\ &= f_{\mathbf{k}}(b, 1) f_{\mathbf{k}}(d, b) + f_{\mathbf{k}}(b^{\perp}, 1) f_{\mathbf{k}}(d, b^{\perp}) \end{aligned}$$

for each $d \in L$. Let $\alpha(b) = 0.2$ and $\alpha'(b) = 0.3$. Then $f_{\mathbf{k}}(b, 1) = 0.29$ and we can write

$$f_{\mathbf{k}}(d, 1) = 0.29 f_{\mathbf{k}}(d, b) + 0.71 f_{\mathbf{k}}(d, b^{\perp}).$$

If we put $d = a$, then

$$f_{\mathbf{k}}(a, b) \in \left[0, \frac{10}{29}\right] \quad \text{and} \quad f_{\mathbf{k}}(a, b^{\perp}) \in \left[0, \frac{10}{71}\right].$$

Therefore

$$0.29 = f_{\mathbf{k}}(b, 1) \neq f_{\mathbf{k}}(b, a) = 0.2,$$

then

b is not independent of a with respect to the state $f_{\mathbf{k}}(., 1)$.

If $f_{\mathbf{k}}(a, 1) = 0.1$, then $f_{\mathbf{k}}(a, 1) = f_{\mathbf{k}}(a, b)$ and so

a is independent of b with respect to the state $f_{\mathbf{k}}(., 1)$ ($a \asymp_{f_{\mathbf{k}}(.,1)} b$).

From the above mentioned it follows that the Boolean algebra as a measurable system $B_1 = \{0, 1, a, a^{\perp}\}$ is independent of the Boolean algebra as a measurable system $B_2 = \{0, 1, b, b^{\perp}\}$ with respect to the conditional state $f_{\mathbf{k}}$ and B_2 is dependent on the B_1 with respect to the conditional state $f_{\mathbf{k}}$. It may be that this approach to the conditional state can help describe some problems of causality in the theory of probability.

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